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MATHEMATICS

Two remarks on diameter 2 properties

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Abstract. A Banach space is said to have the diameter 2 property if the diameter of every nonempty relatively weakly open subset of its unit ball equals 2. In a paper by Abrahamsen, Lima, and Nygaard (Remarks on diameter 2 properties. *J. Conv. Anal.*, 2013, **20**, 439–452), the strong diameter 2 property is introduced and studied. This is the property that the diameter of every convex combination of slices of its unit ball equals 2. It is known that the diameter 2 property is stable by taking ℓ_p-sums for $1 \le p \le \infty$. We show the absence of the strong diameter 2 property on ℓ_p-sums of Banach spaces when $1 < p < \infty$. This confirms the conjecture of Abrahamsen, Lima, and Nygaard that the diameter 2 property and the strong diameter 2 property are different. We also show that the strong diameter 2 property carries over to the whole space from a non-zero M-ideal.

Key words: diameter 2 property, slice, relatively weakly open set.

1. INTRODUCTION

All Banach spaces considered in this note are over the real field. For a Banach space X, its dual space is denoted by X^*, B_X is the closed unit ball of X, and S_X stands for the unit sphere of X. By a *slice* of B_X we mean a set of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

Nygaard and Werner [10] showed that in every infinite-dimensional uniform algebra, every nonempty relatively weakly open subset of its closed unit ball has diameter 2. If a Banach space satisfies this condition, then it is said to have the *diameter* 2 *property* (see, e.g., [1,3,5]).

In addition to the diameter 2 property, Abrahamsen, Lima, and Nygaard [1] consider two other formally different diameter 2 properties – the local diameter 2 property and the strong diameter 2 property.

According to the terminology in [1], a Banach space X has the *local diameter* 2 *property* if every slice of B_X has diameter 2; and X has the *strong diameter* 2 *property* if every convex combination of slices of B_X has diameter 2, i.e., the diameter of $\sum_{i=1}^n \lambda_i S_i$ is 2, whenever $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \ge 0$, with $\sum_{i=1}^n \lambda_i = 1$, and $S_1, \dots, S_n$ are slices of B_X.

The diameter 2 property clearly implies the local diameter 2 property. The strong diameter 2 property implies the diameter 2 property. This follows directly from Bourgain's lemma ([6, Lemma II.1 p. 26]), which asserts that every nonempty relatively weakly open subset of B_X contains some convex combination of slices.

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In this note, we summarize some main results on diameter 2 properties obtained in the Master's Thesis of the second named author. The thesis was defended at the University of Tartu in June 2012.

It is conjectured in [1] that these three diameter 2 properties are different. In Section 2, we will show that there exist Banach spaces with the diameter 2 property but without the strong diameter 2 property. In fact, we prove that the strong diameter 2 property is never stable by taking the ℓ_p-sum for $1 < p < \infty$ (cf. Theorem 1). On the other hand, the diameter 2 property is stable under ℓ_p-sums (see [1, Theorem 3.2]).

The papers [1] and [9] inspired us to consider diameter 2 properties in the context of M-ideals. Section 3 is the result of that study. We show that all three diameter 2 properties carry over to the whole space from a non-zero M-ideal. This generalizes Theorem 3.2 (the case of $p = \infty$) and Proposition 4.6 from [1].

2. STRONG DIAMETER 2 PROPERTY IS NEVER STABLE UNDER ℓ_p-SUMS

Perhaps the most surprising result in [1] is that the local diameter 2 property and the diameter 2 property are stable by taking ℓ_p-sums for $1 < p < \infty$ (see [1, Theorem 3.2]). The same result is true, and even easier also, for $p = 1$ and $p = \infty$. For $p = \infty$, the diameter 2 case was obtained by López Pérez ([9, Lemma 2.1], see also [4, Lemma 2.2]).

One of the questions asked in [1] was whether the strong diameter 2 property is also stable under ℓ_p-sums (see ([1, Question (c)]). The answer was known for $p = 1$ and for $p = \infty$:

- If the Banach spaces X and Y have the strong diameter 2 property, then $X \oplus_1 Y$ has the strong diameter 2 property (see [1, Theorem 2.7 (iii)]). This result is essentially due to Becerra Guerrero and López Pérez in [4, proof of Lemma 2.1 (ii)].
- If a Banach space X has the strong diameter 2 property, then $X \oplus_\infty Y$ has the strong diameter 2 property for any Banach space Y ([1, Proposition 4.6]). We will generalize the last result in Proposition 3.

The following is our main result. It provides an answer, in the negative, to Question (c) in [1]. Moreover, it confirms the conjecture in [1] that the diameter 2 property and the strong diameter 2 property are different.

Theorem 1. *Let X and Y be nontrivial Banach spaces and let $1 < p < \infty$. The Banach space $Z = X \oplus_p Y$ fails the strong diameter 2 property.*

Remark.
(1) Theorem 1 is a joint result with Märt Põldvere.
(2) Theorem 1 was obtained independently by María Acosta, Julio Becerra Guerrero, and Ginés López Pérez; it is included in [2, Theorem 3.2].
(3) Eve Oja has presented another proof of Theorem 1 ([8]).

To prove Theorem 1, we will need the following elementary lemma.

Lemma 2. *Let $1 < p < \infty$ and let q be such that $1/p + 1/q = 1$. If $z^* = (x^*, y^*)$ is an element in $S_{Z^*} = S_{X^* \oplus_q Y^*}$, then for every $\varepsilon > 0$ there exists $\alpha > 0$ such that*

$$\|(\|x\|, \|y\|) - (\|x^*\|^{q-1}, \|y^*\|^{q-1})\|_p < \varepsilon,$$

whenever $z = (x, y)$ is an element in $S(z^*, \alpha)$.

Proof. Note that if $z = (x, y)$ is an element in $S(z^*, \alpha)$, then $(\|x\|, \|y\|)$ and $(\|x^*\|^{q-1}, \|y^*\|^{q-1})$ are both elements of the slice $S((\|x^*\|, \|y^*\|), \alpha)$ of $B_{\ell_p^2}$. Obviously, when α tends to 0, then $\operatorname{diam}(S((\|x^*\|, \|y^*\|), \alpha))$ tends to 0 as well. This proves the result. □

Proof of Theorem 1. In fact, we will show a stronger statement: For every $\lambda \in (0, 1)$, there exists $\alpha, \beta > 0$ and $z^*, \tilde{z}^* \in S_{Z^*}$ such that

$$\lambda S(z^*, \alpha) + (1 - \lambda) S(\tilde{z}^*, \alpha) \subset (1 - \beta) B_Z.$$

Let $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$. We take $z^* = (x^*, 0)$ and $\tilde{z}^* = (0, y^*)$. Then z^* and $\tilde{z}^*$ are elements in S_{Z^*}. Fix $\lambda \in (0, 1)$. Let

$$\varepsilon = 1 - \left(\lambda^p + (1 - \lambda)^p\right)^{1/p}.$$

Clearly, $\varepsilon > 0$. By Lemma 2, there exists $\alpha > 0$ such that

$$\left(\left(\lambda\,\|x\|+(1-\lambda)\,\|\tilde{x}\|\right)^p+\left(\lambda\,\|y\|+(1-\lambda)\,\|\tilde{y}\|\right)^p\right)^{1/p}$$
$$\leq \left(\left(\lambda\cdot 1+(1-\lambda)\cdot 0\right)^p+\left(\lambda\cdot 0+(1-\lambda)\cdot 1\right)^p\right)^{1/p}+\frac{\varepsilon}{2}$$
$$=\left(\lambda^p+(1-\lambda)^p\right)^{1/p}+\frac{\varepsilon}{2}=1-\frac{\varepsilon}{2},$$

whenever $z=(x,y)\in S(z^*,\alpha)$ and $\tilde{z}=(\tilde{x},\tilde{y})\in S(\tilde{z}^*,\alpha)$.

One may take $\beta=\varepsilon/2$. Indeed, for $z=(x,y)\in S(z^*,\alpha)$ and $\tilde{z}=(\tilde{x},\tilde{y})\in S(\tilde{z}^*,\alpha)$, we now have

$$\|\lambda z+(1-\lambda)\tilde{z}\| = \left(\|\lambda x+(1-\lambda)\tilde{x}\|^p+\|\lambda y+(1-\lambda)\tilde{y}\|^p\right)^{1/p}$$
$$\leq \left(\left(\lambda\,\|x\|+(1-\lambda)\,\|\tilde{x}\|\right)^p+\left(\lambda\,\|y\|+(1-\lambda)\,\|\tilde{y}\|\right)^p\right)^{1/p}$$
$$\leq 1-\frac{\varepsilon}{2}. \qquad \square$$

3. DIAMETER 2 PROPERTIES CARRY OVER TO THE WHOLE SPACE FROM A NON-ZERO *M*-IDEAL

We denote the *annihilator* of a subspace Y of a Banach space X by

$$Y^{\perp}=\{x^*\in X^*\colon x^*(y)=0 \quad \text{for all } y\in Y\}.$$

According to the terminology in [7], a closed subspace Y of a Banach space X is called an *M-ideal* if there exists a norm-1 projection P on X^* with $\ker P = Y^{\perp}$ and

$$\|x^*\|=\|Px^*\|+\|x^*-Px^*\| \qquad \text{for all } x^*\in X^*.$$

Relations between M-ideal structure and the diameter 2 property were first considered in [9]. There it is proved that if a proper subspace Y of X is an M-ideal in X and the range of the corresponding projection is 1-norming, then both Y and X have the diameter 2 property (see [9, Theorem 2.4]). In [1, Theorem 4.10] it is shown that, under the same assumptions, one can conclude that both Y and X have even the strong diameter 2 property. An immediate corollary of this is that if a nonreflexive Banach space X is an M-ideal in its bidual, then both X and X^{**} have the strong diameter 2 property.

One cannot omit the assumption that the range of the corresponding projection is 1-norming. To see an example of this, let Y be any Banach space and let $X = Y\oplus_\infty c_0$. Then, by [1, Proposition 4.6] (or Proposition 3 below), X has the strong diameter 2 property and Y is an M-ideal in X.

In the following we will show that if a non-zero M-ideal Y has some diameter 2 property, then X has the same diameter 2 property without the assumption that the range of the projection is 1-norming. This, at the same time, generalizes Theorem 3.2 (the case of $p=\infty$) and the above-mentioned Proposition 4.6 of [1].

Proposition 3. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the strong diameter 2 property, then X has the strong diameter 2 property.*

Proof. Let $\sum_{i=1}^n \lambda_i S(x_i^*, \alpha_i)$ be a convex combination of slices of B_X, where $n \in \mathbb{N}$, and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$. Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\alpha_1, \dots, \alpha_n\}/3$.

We will show the existence of $x_1^1, \dots, x_n^1, x_1^2, \dots, x_n^2 \in B_X$ such that $x_i^k \in S(x_i^*, \alpha_i)$ for every $i = 1, \dots, n$, $k = 1, 2$, and

$$\left\| \sum_{i=1}^n \lambda_i (x_i^1 - x_i^2) \right\| > \frac{2-\varepsilon}{1+\varepsilon}.$$

Denote by P the M-ideal projection on X^* with $\ker P = Y^\perp$. For every $i = 1, \dots, n$, we take

$$y_i^* = \frac{Px_i^*}{\|Px_i^*\|} \qquad \text{and} \qquad \beta_i = \frac{\varepsilon - \varepsilon \|Px_i^*\| + \varepsilon^2}{\|Px_i^*\|}.$$

Note that, if $Px_i^* \neq 0$, then $\beta_i > 0$. If $Px_i^* = 0$, we can take $y_i^* \in S_{Y^*}$ and $\beta_i > 0$ to be arbitrary. Observe that $\sum_{i=1}^n \lambda_i S(y_i^*, \beta_i)$ is a convex combination of slices of B_Y. Since Y has the strong diameter 2 property, we can find $y_1^1, \dots, y_n^1$ and $y_1^2, \dots, y_n^2$ in B_Y such that

$$Px_i^*(y_i^k) > (\|Px_i^*\| - \varepsilon)(1+\varepsilon), \qquad k = 1, 2, \quad i = 1, \dots, n,$$

and

$$\left\| \sum_{i=1}^n \lambda_i (y_i^1 - y_i^2) \right\| > 2 - \varepsilon.$$

There are $x_1, \dots, x_n \in B_X$ such that

$$(x_i^* - Px_i^*)(x_i) > (\|x_i^* - Px_i^*\| - \varepsilon)(1+\varepsilon)$$

for every $i = 1, \dots, n$.

Since Y is an M-ideal in X, then by [11, Proposition 2.3], we can, for every $i = 1, \dots, n$, choose $z_i \in B_Y$ such that

$$\|y_i^k + x_i - z_i\| < 1 + \varepsilon, \qquad k = 1, 2,$$

and

$$|Px_i^*(x_i - z_i)| < \varepsilon.$$

We take

$$x_i^k = \frac{y_i^k + x_i - z_i}{1+\varepsilon}, \qquad k = 1, 2, \quad i = 1, \dots, n.$$

Now, for every $i = 1, \dots, n$, for every $k = 1, 2$, x_i^k is an element in $S(x_i^*, \alpha_i)$, because

$$\begin{aligned} x_i^*(x_i^k) &= \frac{x_i^*(y_i^k + x_i - z_i)}{1+\varepsilon} \\ &= \frac{Px_i^*(y_i^k) + (x_i^* - Px_i^*)(x_i) + Px_i^*(x_i - z_i)}{1+\varepsilon} \\ &> \|Px_i^*\| - \varepsilon + \|x_i^* - Px_i^*\| - \varepsilon - \varepsilon \\ &= \|x_i^*\| - 3\varepsilon > 1 - \alpha_i. \end{aligned}$$

Finally, observe that

$$\left\| \sum_{i=1}^n \lambda_i (x_i^1 - x_i^2) \right\| = \frac{1}{1+\varepsilon} \left\| \sum_{i=1}^n \lambda_i (y_i^1 - y_i^2) \right\| > \frac{2-\varepsilon}{1+\varepsilon}. \qquad \square$$

We conclude our study with the local diameter 2 and the diameter 2 versions of Proposition 3.

Proposition 4. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the local diameter 2 property, then X has the local diameter 2 property.*

Proof. Take $n = 1$ in the proof of Proposition 3. □

The next result is obtained in the proof of [9, Theorem 2.4], but not stated explicitly. We will give a direct proof of this result.

Proposition 5. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the diameter 2 property, then X has the diameter 2 property.*

Proof. The proof is similar to the proof of Proposition 3.

Let U be a nonempty relatively weakly open subset of B_X containing an element x_0. We may assume that

$$\{x \in B_X :\ |x_i^*(x - x_0)| < \gamma, \quad i = 1, \ldots, n\} \subset U,$$

for some $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in S_{X^*}$, and $\gamma > 0$.

Denote by P the M-ideal projection on X^* with $\ker P = Y^\perp$, and let $\delta = \max\{\|Px_i^*\| : i = 1, \ldots, n\}$. Let $\varepsilon > 0$ be such that $\varepsilon(4 + \delta) < \gamma$. We will show the existence of elements x and $\tilde{x}$ in U such that

$$\|x - \tilde{x}\| > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

Since B_Y is dense in B_X in the weak topology $\sigma(X, \operatorname{ran} P)$, we can find an element $y_0 \in B_Y$ such that

$$|Px_i^*(x_0 - y_0)| < \varepsilon$$

for every $i = 1, \ldots, n$. Consider the set

$$V = \{y \in B_Y :\ |Px_i^*(y - y_0)| < \varepsilon(\delta + 1), \quad i = 1, \ldots, n\}.$$

Clearly V is a nonempty relatively weakly open subset of B_Y. By the assumption, there are $y_1, y_2 \in V$ with $\|y_1 - y_2\| > 2 - \varepsilon$.

Since Y is an M-ideal in X, by [11, Proposition 2.3], there is an element $z_0 \in B_Y$ such that

$$\|y_k + x_0 - z_0\| < 1 + \varepsilon, \qquad k = 1, 2,$$

and

$$|Px_i^*(x_0 - z_0)| < \varepsilon$$

for every $i = 1, \ldots, n$.

We take

$$x_1 = \frac{y_1 + x_0 - z_0}{1 + \varepsilon} \qquad \text{and} \qquad x_2 = \frac{y_2 + x_0 - z_0}{1 + \varepsilon}.$$

Now, for every $i = 1, \ldots, n$, we have

$$\begin{aligned}
|x_i^*(x_1 - x_0)| &= \frac{1}{1 + \varepsilon} \left|x_i^*(y_1 - \varepsilon x_0 - z_0) \pm Px_i^*(x_0) \pm Px_i^*(y_0)\right| \\
&\le \frac{1}{1 + \varepsilon} \Big(|Px_i^*(y_1 - y_0)| + |Px_i^*(x_0 - z_0)| + \varepsilon\, |x_i^*(x_0)| + |Px_i^*(y_0 - x_0)| \Big) \\
&< \frac{1}{1 + \varepsilon} (\varepsilon\delta + 4\varepsilon) < \gamma.
\end{aligned}$$

Thus, $x_1 \in U$. Similarly one can show that $x_2 \in U$. Finally, observe that

$$\|x_1 - x_2\| = \frac{1}{1 + \varepsilon}\, \|y_1 - y_2\| > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

□

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Kaks märkust diameeter-2 omaduste kohta

Rainis Haller ja Johann Langemets

On tõestatud, et artiklis [1] vaadeldud diameeter-2 omadus ja tugev diameeter-2 omadus on erinevad. On näidatud, kuidas diameeter-2 omadused kanduvad M-ideaalilt kogu ruumile.