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Nonlinear dispersive wave equations for microstructured solids

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Abstract. Dispersion is a characteristic feature for wave propagation in microstructured solids. In the case of linear elasticity, dispersion effects are modelled by higher-order derivatives included into the wave equation. Nonlinear effects are also well known in wave propagation in solids. In principle, such effects may appear at the macroscale as well as at the microscale. The microstructural influence is often taken into account by the introduction of internal variables. This suggests that internal variables may behave nonlinearly. It is shown that the nonlinear behaviour of internal variables may lead at the macroscale to the Benjamin–Bona–Mahoney equation or the Camassa–Holm equation.

Key words: dispersive wave equations, microstructured solids, nonlinearity.

1. INTRODUCTION

Wave propagation in a homogeneous medium is a well described phenomenon if the linear elasticity theory is applicable [1,2]. The situation becomes more complicated if the medium is inhomogeneous. Modern advanced materials (composites, functionally graded materials, shape memory alloys, etc.), nowadays widely used in engineering, are inhomogeneous by definition. Properties of such materials might not be primarily controlled by their chemical composition but rather by their microstructure. Microstructure is usually characterized by a length scale, which is much smaller than the length scale of the element. Nevertheless, the influence of the microstructure may not be necessarily small, especially in dynamics. The effect of inhomogeneity manifests itself in slowing down of the propagation and in the dispersion of the wave.

In the linear case, classical theory is still sufficient for the description of wave propagation if we know all the details of a given microstructure, namely, size, shape, composition, location, and properties of inclusions as well as properties of the carrier medium. Typically, however, our knowledge about the microstructure is limited: we know usually only the characteristic scale of the microstructure and, possibly, physical properties of inclusions. That is why several modifications of the linear wave equation have been proposed to describe microstructural influence on wave propagation in heterogeneous materials [3–9]. The review and generalization of such linear models has been presented recently [10,11].

Even more complicated behaviour of waves in solids is observed if nonlinear effects enter into the play [5,12]. The main attention is paid usually for travelling solitary wave solution of the corresponding nonlinear dispersive-dissipative Korteveg–de Vries-type equation [5,12–19]. It should be noted that similar solution possesses the so-called Benjamin–Bona–Mahoney equation and the Camassa–Holm equation derived for water waves [20–22]. A natural question is the following: could similar equations apply for wave propagation in solids?

The answer is ”yes”, at least if we invoke the extended internal variable approach [23] for the description of microstructural effects. In the framework of the dual internal variable theory [24], a fully coupled system of equations for macro-motion and microstructure evolution is represented in the form of conservation laws. Nonlinearities may be presented at both macro- and micro-scales. In the latter case, the modelling of the nonlinear behaviour of internal variables may lead to the Benjamin–Bona–Mahoney equation or the Camassa–Holm equation at the macroscale.

2. BALANCE LAWS IN THE MATERIAL FORMULATION

The existence of the microstructure generally means that the medium is inhomogeneous. Therefore, we apply the canonical form of balance equations [25], where the inhomogeneities are treated in the most consistent way.

In the case of the thermoelastic conductors of heat, one-dimensional motion is governed by local balance laws for linear momentum and energy (no body forces)

$$(\rho_0 v)_t - \sigma_x = 0, \tag{1}$$

$$\left(\rho_0 v^2/2 + E\right)_t - (\sigma v - Q)_x = 0, \tag{2}$$

and by the second law of thermodynamics

$$S_t + (Q/\theta + K)_x \geq 0. \tag{3}$$

Here t is time, ρ_0 is the matter density, $v = u_t$ is the physical velocity, σ is the Cauchy stress, E is the internal energy per unit volume, S is the entropy per unit volume, θ is temperature, Q is the material heat flux, and subscript denotes derivatives. The “extra entropy flux” K vanishes in most cases, but this is not a basic requirement.

2.1. Material form of the energy conservation

The canonical energy equation is obtained by introducing the free energy per unit volume $W := E - S\theta$ and taking into account the balance of linear momentum (1)

$$(S\theta)_t + Q_x = h^{int}, \quad h^{int} := \sigma\varepsilon_t - W_t, \tag{4}$$

where the right-hand side is formally an internal heat source [26].

In the case of non-zero extra entropy flux, the second law of thermodynamics gives

$$-(W_t + S\theta_t) + \sigma\varepsilon_t + (\theta K)_x - (Q/\theta + K)\,\theta_x \geq 0, \tag{5}$$

where $\varepsilon = u_x$ is the one-dimensional strain measure.

2.2. Material momentum conservation

Multiplying Eq. (1) by u_x, Eq. (1) yields the following material balance of momentum (cf. [25])

$$P_t - b_x = f^{int} + f^{inh}, \tag{6}$$

where the *material momentum* P, the material Eshelby stress b, the material inhomogeneity force f^{inh}, and the material internal force f^{int} are defined by [25]

$$P := -\rho_0 u_t u_x, \quad b := -\left(\rho_0 v^2/2 - W + \sigma\varepsilon\right), \tag{7}$$

$$f^{inh} := (\rho_0)_x v^2/2 - W_x|_{expl}\,, \quad f^{int} := \sigma u_{xx} - W_x|_{impl}\,. \tag{8}$$

Here the subscript notations *expl* and *impl* mean the derivative keeping the fields fixed (and thus extracting the explicit dependence on x), and taking the derivative only through the fields present in the function, respectively. The canonical equations for energy and momentum (4) and (6) are the most general expressions we can write down without a postulate of the full dependence of the free energy W [26].

3. INTERNAL VARIABLES

In the framework of the phenomenological continuum theory it is assumed that the influence of the microstructure on the overall macroscopic behaviour can be taken into account by the introduction of an internal variable φ, which we associate with the integral distributed effect of the microstructure [23]. Following the concept of dual internal variables [24], we introduce an additional auxiliary internal variable ψ. Then the free energy W is specified as the general sufficiently regular function of the strain, temperature, internal variables φ, ψ and their space derivatives

$$W = \overline{W}(u_x, \theta, \varphi, \varphi_x, \psi, \psi_x). \tag{9}$$

The corresponding equations of state are given by

$$\sigma := \frac{\partial \overline{W}}{\partial u_x}, \quad S := -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \quad \xi := -\frac{\partial \overline{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \psi_x}. \tag{10}$$

The accepted functional dependence (9) and the equations of state (10) lead to the representation of the internal force (8) in the form

$$\begin{aligned} f^{int} := \sigma u_{xx} - W_x|_{impl} &= -\frac{\partial \overline{W}}{\partial \theta}\frac{\partial \theta}{\partial x} - \frac{\partial \overline{W}}{\partial \varphi}\frac{\partial \varphi}{\partial x} - \frac{\partial \overline{W}}{\partial \varphi_x}\frac{\partial \varphi_x}{\partial x} - \frac{\partial \overline{W}}{\partial \psi}\frac{\partial \psi}{\partial x} - \frac{\partial \overline{W}}{\partial \psi_x}\frac{\partial \psi_x}{\partial x} \\ &= S\theta_x + \tau\varphi_x + \eta\,\varphi_{xx} + \xi\,\psi_x + \zeta\,\psi_{xx} = f^{th} + f^{intr}. \end{aligned} \tag{11}$$

Accordingly, the internal heat source h^{int} is calculated as follows:

$$\begin{aligned} h^{int} := \sigma\dot{\varepsilon} - \frac{\partial W}{\partial t} &= -\frac{\partial \overline{W}}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial \overline{W}}{\partial \varphi}\frac{\partial \varphi}{\partial t} - \frac{\partial \overline{W}}{\partial \varphi_x}\frac{\partial \varphi_x}{\partial t} - \frac{\partial \overline{W}}{\partial \psi}\frac{\partial \psi}{\partial t} - \frac{\partial \overline{W}}{\partial \psi_x}\frac{\partial \psi_x}{\partial t} \\ &= S\theta_t + \tau\varphi_t + \eta\,\varphi_{xt} + \xi\,\psi_t + \zeta\,\psi_{xt} = h^{th} + h^{intr}. \end{aligned} \tag{12}$$

Here the introduced thermal source terms are defined in terms of space and time derivatives of temperature

$$f^{th} := S\theta_x, \quad h^{th} := S\theta_t, \tag{13}$$

whereas "intrinsic" source terms are determined by the internal variables

$$f^{intr} := \tau\varphi_x + \eta\,\varphi_{xx} + \xi\,\psi_x + \zeta\,\psi_{xx}, \quad h^{intr} := \tau\varphi_t + \eta\,\varphi_{xt} + \xi\,\psi_t + \zeta\,\psi_{xt}. \tag{14}$$

Accounting for the expression of the internal heat source (12), the dissipation inequality (5) can be rewritten as

$$\Phi = \tau\varphi_t + \eta\varphi_{xt} + \xi\psi_t + \zeta\psi_{xt} - (Q/\theta + K)\theta_x + (\theta K)_x \geq 0. \tag{15}$$

We rearrange the dissipation inequality by adding and subtracting the same terms

$$\Phi = \tau\varphi_t - \eta_x\varphi_t + \eta_x\varphi_t + \eta\varphi_{xt} + \xi\psi_t - \zeta_x\psi_t + \zeta_x\psi_t + \zeta\psi_{xt} - (Q/\theta + K)\theta_x + (\theta K)_x \geq 0, \tag{16}$$

and then arrive at

$$\Phi = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t - (Q/\theta + K)\theta_x + (\eta\varphi_t + \zeta\psi_t + \theta K)_x \geq 0. \tag{17}$$

Following the scheme, originally developed in [27] for materials with diffusive dissipative processes, described by means of internal variables of state, we choose the non-zero extra entropy flux K in the form

$$K = -\theta^{-1}\eta\varphi_t - \theta^{-1}\zeta\psi_t. \tag{18}$$

Such a choice allows us to reduce the dissipation inequality to

$$\Phi = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t - \left(\frac{Q - \eta\varphi_t - \zeta\psi_t}{\theta}\right)\theta_x \geq 0. \tag{19}$$

In this case, the dissipation is clearly decomposed into intrinsic and thermal parts. The latter means that the dissipation inequality in the isothermal case reduces to

$$\Phi = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t \geq 0. \tag{20}$$

Evolution equations for internal variables follow from the general solution of the dissipation inequality (20), which has the form [28]

$$\begin{pmatrix} \varphi_t \\ \psi_t \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \tau - \eta_x \\ \xi - \zeta_x \end{pmatrix}, \tag{21}$$

where coefficients of matrix L are dependent on state variables.

However, in the case of zero dissipation, Eq. (20) yields that the evolution equations for internal variables can be represented in the form

$$\varphi_t = R(\xi - \zeta_x), \qquad \psi_t = -R(\tau - \eta_x), \tag{22}$$

where R is an arbitrary coefficient.

4. CONSTITUTIVE MODEL

Having the evolution equations for internal variables in the non-dissipative case, we can construct a microstructure model. We begin with the free energy dependence in the form

$$\overline{W} = \frac{\rho_0 c^2}{2}u_x^2 + Au_x\varphi + \tilde{A}u_x\varphi_x + au_x\left(\frac{dF(u)}{du}\right)_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{2}D\psi^2 + bG(\varphi_x), \tag{23}$$

where c is the elastic wave speed, $A, \tilde{A}, B, C$, and D are material parameters, $F(u)$ and $G(\varphi_x)$ are nonlinear contributions at macro- and microscale, respectively, a and b are scaling coefficients. For simplicity, we include only the contribution of the second internal variable itself. In this case, stresses are calculated as follows:

$$\begin{aligned} \sigma &= \frac{\partial \overline{W}}{\partial u_x} = \rho_0 c^2 u_x + A\varphi + \tilde{A}\varphi_x + a\left(\frac{dF(u)}{du}\right)_x, \\ \eta &= -\frac{\partial \overline{W}}{\partial \varphi_x} = -\tilde{A}u_x - C\varphi_x - bG', \quad \zeta = -\frac{\partial \overline{W}}{\partial \psi_x} = 0. \end{aligned} \tag{24}$$

The interactive internal forces τ and ξ are, respectively,

$$\tau = -\frac{\partial \overline{W}}{\partial \varphi} = -Au_x - B\varphi, \quad \xi = -\frac{\partial \overline{W}}{\partial \psi} = -D\psi. \tag{25}$$

The evolution equations (22) in the case of zero dissipation take the form

$$\varphi_t = R(\xi - \zeta_x) = -RD\psi, \tag{26}$$

$$\psi_t = -R(\tau - \eta_x) = R(Au_x + B\varphi - \tilde{A}u_{xx} - C\varphi_{xx} - bG'_x). \tag{27}$$

It follows immediately from Eqs (26) and (27) that the evolution equation for the primary internal variable (26) can be rewritten as the hyperbolic equation

$$\varphi_{tt} = R^2 D(\tau - \eta_x). \tag{28}$$

Accordingly, the balance of linear momentum results in

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A\varphi_x + \tilde{A}\varphi_{xx} + a[F'(u)]_{xx}, \tag{29}$$

and the evolution equation for the primary internal variable gives

$$I\varphi_{tt} = C\varphi_{xx} + \tilde{A}u_{xx} + bG'_x - Au_x - B\varphi, \tag{30}$$

where $I = 1/(R^2 D)$ is an internal inertia measure.

4.1. Single dispersive wave equation

To derive the single equation we make following steps. We determine the first derivative of the internal variable from Eq. (30)

$$B\varphi_x = -I\varphi_{ttx} + C\varphi_{xxx} + \tilde{A}u_{xxx} + bG'_{xx} - Au_{xx}. \tag{31}$$

The third mixed derivative φ_{ttx} follows from Eq. (29)

$$A\varphi_{ttx} = \left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx} - a[F'(u)]_{xx}\right)_{tt} - \tilde{A}\varphi_{ttxx}. \tag{32}$$

The appeared fourth-order mixed derivative of the internal variable is calculated by means Eq. (30)

$$I\varphi_{ttxx} = C\varphi_{xxxx} + \tilde{A}u_{xxxx} + bG'_{xxx} - Au_{xxx} - B\varphi_{xx}, \tag{33}$$

and, in its turn, the fourth-order space derivative is determined again from Eq. (29)

$$\tilde{A}\varphi_{xxxx} = \left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx} - a[F'(u)]_{xx}\right)_{xx} - A\varphi_{xxx}. \tag{34}$$

Collecting all the results, Eqs (31)–(34), and substituting them into the balance of linear momentum (29), we arrive at the dispersive wave equation in the form

$$\begin{aligned} \rho_0 u_{tt} - \rho_0 c^2 u_{xx} - a[F'(u)]_{xx} =& \frac{C}{B}\left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx} - a[F'(u)]_{xx}\right)_{xx} \\ & - \frac{I}{B}\left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx} - a[F'(u)]_{xx}\right)_{tt} + \frac{\tilde{A}^2}{B}u_{xxxx} + \frac{\tilde{A}b}{B}G'_{xxx} - \frac{A^2}{B}u_{xx} + \frac{Ab}{B}G'_{xx}. \end{aligned} \tag{35}$$

5. EXAMPLES OF NONLINEAR DISPERSIVE WAVE EQUATIONS

5.1. The Boussinesq equation

The obtained dispersive wave equation can be reduced to the Boussinesq equation under following assumptions:
(1) $I = 0$, which means zero microinertia;
(2) $G = 0$ that corresponds to the absence of nonlinearity in the microstructure;
(3) $A = 0$ (no coupling between strain and internal variable; only gradients are coupled).

As a result, Eq. (35) reduces to

$$\rho_0 u_{tt} - \rho_0 c^2 u_{xx} - a[F'(u)]_{xx} = \frac{C}{B}\left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx} - a[F'(u)]_{xx}\right)_{xx} + \frac{\tilde{A}^2}{B} u_{xxxx}. \tag{36}$$

This equation belongs to the so-called "Boussinesq paradigm" [16]. The original Boussinesq equation needs further simplifications. In particular, it should be suggested that $C = 0$, which means that the internal variable φ is equal to the strain gradient. The nonlinearity function should be chosen as $F(u) = u^3$ [16], which yields

$$u_{tt} - c^2 u_{xx} = \left(\frac{3au^2}{\rho_0} + \frac{\tilde{A}^2}{\rho_0 B} u_{xx}\right)_{xx}. \tag{37}$$

The original Boussinesq equation can be recovered by identification u with free surface elevation and choosing coefficients as $c^2 = gh$, $a = g/2$, $\tilde{A}^2/\rho_0 B = gh^3/3$ (g is the acceleration by gravity, h is the constant mean depth)

$$u_{tt} - ghu_{xx} = \left(\frac{3g}{2}u^2 + \frac{gh^3}{3} u_{xx}\right)_{xx}. \tag{38}$$

It is well known that the Boussinesq equation (38) describes waves, which can propagate both to the right and to the left (the two-way long-wave equation).

5.2. The Korteveg–de Vries equation

The Korteveg–de Vries equation describes the time evolution of the wave propagating in one direction. To derive the Korteveg–de Vries equation from the Boussinesq equation, we represent the Boussinesq equation (38) in the normalized form

$$u_{tt} - u_{xx} = \varepsilon\left(u^2 + u_{xx}\right)_{xx}. \tag{39}$$

Introducing the new variable v by $u = v_x$, after one integration we obtain

$$v_{tt} - v_{xx} = \varepsilon\left(v_x^2 + v_{xxx}\right)_x. \tag{40}$$

Representing v in the form of an asymptotic multiple-scale expansion

$$v(x,t) = f(\xi, T) + \varepsilon v^1(x,t) + \ldots, \tag{41}$$

where $\xi = x - t, T = \varepsilon t$, we obtain

$$v_t = f_\xi \xi_t + f_T T_t + \varepsilon v_t^1 + \ldots = -f_\xi + \varepsilon f_T + \varepsilon v_t^1 + \ldots. \tag{42}$$

Accordingly,

$$\begin{gathered} v_{tt} = -f_{\xi\xi}\xi_t - f_{\xi T} T_t + \varepsilon f_{\xi T}\xi_t + \varepsilon f_{TT} T_t + \varepsilon v_{tt}^1 + \ldots = f_{\xi\xi} - 2\varepsilon f_{\xi T} + \varepsilon^2 f_{TT} + \varepsilon v_{tt}^1 + \ldots \\ v_{xx} = f_{\xi\xi} + \varepsilon v_{xx}^1 + \ldots. \end{gathered} \tag{43}$$

Inserting expressions (41)–(43) into Eq. (40), we arrive at

$$v^1_{tt} - v^1_{xx} = 2f_{\xi T} + 2f_\xi f_{\xi\xi} + f_{\xi\xi\xi\xi} + \dots . \tag{44}$$

The function v^1 will grow linearly in $x+t$, unless

$$2f_{\xi T} + 2f_\xi f_{\xi\xi} + f_{\xi\xi\xi\xi} = 0. \tag{45}$$

The last equation reduces to the canonical Korteveg–de Vries equation by setting $q = f_\xi/3, \tau = T/2$

$$q_\tau + 6qq_\xi + q_{\xi\xi\xi} = 0. \tag{46}$$

We can apply a similar asymptotic procedure to some other particular cases of the dispersive wave equation (35).

5.3. The Benjamin–Bona–Mahoney equation

As an example, we consider first the normalized reduction of Eq. (35)

$$u_{tt} - u_{xx} - \varepsilon(u^2)_{xx} = \left(u_{tt} - u_{xx} - \varepsilon(u^2)_{xx}\right)_{tt} + 2\varepsilon u_{xx} + \varepsilon\tilde{G}'_{xx}, \tag{47}$$

which corresponds to following assumptions:
(1) $C = 0$;
(2) $\tilde{A} = 0$ (no coupling between strain and gradients of internal variable);
(3) $F(u) = \varepsilon u^3/3$ (similar to that in the Boussinesq equation).

The nonlinearity term $\tilde{G}$ is chosen to compensate the macroscopic nonlinearity in the right hand side of Eq. (47):

$$\tilde{G}' = (u^2)_{tt} \quad \text{or} \quad \tilde{G}' = (u^2)_{xx}.$$

Both choices are equivalent in the first-order approximation and suggest the assumption $\varphi_x \sim u$. Then Eq. (47) is simplified to

$$u_{tt} - u_{xx} - \varepsilon(u^2)_{xx} = (u_{tt} - u_{xx})_{tt} + 2\varepsilon u_{xx}. \tag{48}$$

Again, we introduce the variable v, which gives (after one integration)

$$v_{tt} - v_{xx} - \varepsilon(v_x^2)_x = (v_{tt} - v_{xx})_{tt} + 2\varepsilon v_{xx}. \tag{49}$$

Using Eqs (41)–(43), we obtain the following equation

$$2f_{\xi T} + 2f_\xi f_{\xi\xi} - 2f_{\xi\xi\xi T} + 2f_{\xi\xi} = 0, \tag{50}$$

which can be reduced to the Benjamin–Bona–Mahoney equation [20] by setting $q = f_\xi$

$$q_T + q_\xi + qq_\xi - q_{\xi\xi T} = 0. \tag{51}$$

This equation is sometimes called the regularized long-wave equation.

5.4. The Camassa–Holm equation

The Camassa–Holm equation can be recovered by a generalization of the previous case. We start with

$$u_{tt} - u_{xx} - \varepsilon(3u^2)_{xx} = (u_{tt} - u_{xx})_{tt} + 2k\varepsilon u_{xx} + \varepsilon(u_x^2 + 2uu_{xx})_{xx}, \tag{52}$$

which means that nonlinearity term $\tilde{G}'$ takes the form

$$\tilde{G}' = (u_x^2 + 2uu_{xx} - (u^2)_{tt}).$$

Again, it is possible if $\varphi_x \sim u$. In terms of the variable v Eq. (52) is represented as

$$v_{tt} - v_{xx} - \varepsilon(3v_x^2)_x = -(v_{tt} - v_{xx})_{tt} + 2k\varepsilon v_{xx} + \varepsilon(v_{xx}^2 + 2v_x v_{xxx})_x. \tag{53}$$

The equation for the leading term in the multiple-scale asymptotic expansion takes the form

$$2f_{\xi T} + 6f_\xi f_{\xi\xi} - 2f_{\xi\xi\xi T} + 4kf_{\xi\xi} = 4f_{\xi\xi} f_{\xi\xi\xi} + 2f_\xi f_{\xi\xi\xi\xi}, \tag{54}$$

and can be reduced to the Camassa–Holm equation [21] by setting $q = f_\xi$

$$q_T + 2kq_\xi + 3qq_\xi - q_{\xi\xi T} = 2q_\xi q_{\xi\xi} + qq_{\xi\xi\xi}. \tag{55}$$

This equation incorporates nonlinear dispersive terms in addition to those terms associated with the Benjamin–Bona–Mahoney equation.

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Mittelineaarsed lainevõrrandid mikrostruktuursete tahkiste jaoks

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Dispersioon on üks lainelevi mõjutav tahkise iseloomulik omadus. Lineaarsel juhul modelleeritakse dispersiivsete efektide mõju lainevõrrandis sisalduvate kõrgemat järku tuletiste abil. Teiseks tuntud lainelevi mõjutavaks efektiks on mittelineaarsus, mis võib ilmneda nii mikro- kui ka makroskaalas. Mikrostruktuuri mõju kirjeldamiseks tuuakse tihti sisse sisemuutujad. Käesolevas artiklis on näidatud, et mittelineaarsel juhul võib sisemuutujate kasutamine mikroskaalal anda makroskaalal tulemuseks Benjamini-Bona-Mahoney või Camassa-Holmi võrrandi.